

# Girth Alternative for Subgroups of $PL_o(I)$

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**ABSTRACT:** We prove the Girth Alternative for finitely generated subgroups of  $PL_o(I)$ . We also prove that a finitely generated subgroup of  $Homeo(I)$  which is sufficiently rich with hyperbolic-like elements has infinite girth.

The notion of a girth for a finitely generated group was first introduced in [S] motivated by the study of Heegaard splittings of closed 3-manifolds.

**Definition 1.** Let  $\Gamma$  be a finitely generated group. For any finite generating set  $S$  of  $\Gamma$ ,  $girth(\Gamma, S)$  will denote the minimal length of relations among the elements of  $S$ . Then we define  $girth(\Gamma) = \sup_{\langle S \rangle = \Gamma, |S| < \infty} girth(\Gamma, S)$ .

By definition above, an infinite cyclic group has infinite girth, but this fact should be viewed as a degeneracy since (as remarked in [Ak1]) any group satisfying a law and non-isomorphic to  $\mathbb{Z}$  has a finite girth.

In [Ak2], we have proved that if a finitely generated group is word hyperbolic, or one-relator, or linear then it is either virtually solvable or has infinite girth. More generally, given a class  $C$  of finitely generated groups, we will say that  $C$  satisfies Girth Alternative if any group from the class  $C$  is either virtually solvable or has infinite girth.

In this note, we will prove that the Girth Alternative holds for subgroups of  $PL_o(I)$  - the group of orientation preserving piecewise linear homeomorphisms of the closed interval. It is known that any virtually solvable subgroup of  $PL_o(I)$  is indeed solvable (see [Bl], Corollary 1.3.) so the Girth Alternative in this case is equivalent to the following

**Theorem 1.** Any finitely generated subgroup of  $PL_o(I)$  is either solvable or has infinite girth.

It is easy to prove the Girth Alternative for  $Diff_\omega(I)$  - the group of orientation preserving analytic diffeomorphism of  $I$ , however, we do not know if the alternative still holds when the regularity is decreased. The following questions are interesting to us:

**Question 1.** Does Girth Alternative hold for subgroups

a) of  $Homeo_+(I)$ ?      b) of  $Diff_+(I)$ ?

**Question 2.** Is there a finitely generated subgroup of  $PL_o(I)$  which is not embeddable into  $Diff_+(I)$ ?

In the proof of Theorem 1, as a crucial tool, we use the result of C.Bleak on the existence of arbitrarily high towers in a non-solvable subgroup of  $PL_o(I)$  ([Bl]). The following notions have been borrowed from [Bl] (we use slightly different terminology; our definition of a tower differs from the one in [Bl]):

**Definition 2.** An ordered  $n$ -tuple  $(f_1, \dots, f_n)$  of elements of  $PL_o(I)$  is said to form a tower if there exist intervals  $(a_i, b_i)$ ,  $1 \leq i \leq n$  such that

- (i)  $0 < a_1 < \dots < a_n < b_n < \dots < b_1 < 1$ ;
- (ii) for all  $i \in \{1, \dots, n\}$ ,  $f_i(a_i) = a_i$ ,  $f(b_i) = b_i$ , and  $f_i$  has finitely many fixed points in  $(a_i, b_i)$ .
- (iii) for all  $i, j \in \{1, \dots, n\}$ , if  $i < j$  then  $f_i(x) > f_j(x)$ ,  $\forall x \in [a_j, b_j]$

We will denote the tower by  $T = [(f_1, \dots, f_n); (a_1, b_1), \dots, (a_n, b_n)]$ ;  $n$  will be called the height of the tower  $T$ , and the interval  $(a_i, b_i)$  will be called the  $i$ -th base of the tower.

**Definition 3.** We will say that a tower  $T = [(f_1, \dots, f_n); (a_1, b_1), \dots, (a_n, b_n)]$  is suitable if for any nonzero integer  $p$  and for all  $1 \leq i < j \leq n$ , the following two conditions hold

- 1)  $f_i^p([a_j, b_j]) \cap [a_j, b_j] = \emptyset$ .
- 2)  $f_i^p([a_j, b_j]) \cap \cup_{i+1 \leq k \leq n} \text{supp}(f_k) = \emptyset$ .

**Remark 1.** If the  $n$ -tuple  $(f_1, \dots, f_n)$  of elements of  $PL_o(I)$  forms a tower then for sufficiently big  $q \in \mathbb{N}$ , the  $n$ -tuple  $(f_1^q, \dots, f_n^q)$  forms a tower with the same bases which satisfies condition 1) of Definition 3. Also, existence of a suitable tower of arbitrary height in non-solvable subgroups of  $PL_o(I)$  follows from the existence of the *exemplary towers* of arbitrary height (see [Bl] for the definition of exemplary towers).

**Proof of Theorem 1.** For any natural number  $r$ , let  $G_r = \mathbb{Z} \wr (\mathbb{Z} \wr (\mathbb{Z} \wr \dots (\mathbb{Z} \wr \mathbb{Z}) \dots))$  where the iterated wreath product is taken  $r$  times. The group  $G_r$  can be defined inductively as  $G_0 = 1$ ,  $G_{i+1} = \mathbb{Z} \wr G_i$ ,  $0 \leq i \leq r-1$ . In the wreath product  $\mathbb{Z} \wr G_i$ , the standard generator of the acting group  $\mathbb{Z}$  will be denoted by  $g_{i+1}$ . (in [Bl], the group  $G_r$  is denoted by  $W_r$ ).

The following lemma will be useful and follows from the proof of Lemma 2.3. in [Ak1].

**Lemma 1.** For all  $q, k \in \mathbb{N}$ , there exist  $r \in \mathbb{N}$  and  $w_1, \dots, w_k \in G_r$  such that there is no relation of length less than  $q$  among  $w_1, \dots, w_k$ .  $\square$

Let  $m$  be a positive integer, and  $\Gamma$  be a non-solvable group generated by a finite subset  $S = \{\gamma_1, \dots, \gamma_s\}$ . Without loss of generality we may assume that  $\text{supp } \Gamma := \cup_{\gamma \in \Gamma} \text{supp } \gamma = [0, 1]$ .

We will fix the left-invariant Cayley metric on  $\Gamma$  with respect to  $S$  and denote it by  $|\cdot|$ . Let  $B_m(1)$  denotes the ball of radius  $m$  centered at identity element  $1 \in \Gamma$ , and  $S_m(\Gamma, S, c) = \{\gamma \in B_m(1) \mid \gamma'(0) = c\}$ ,  $S_m(\Gamma, S) = \{\gamma \in B_m(1) \mid \gamma'(0) = 1\}$ .

Let  $q = 2m^2$ . By Lemma 1, there exists  $r \in \mathbb{N}$  such that in the group  $G_r$  there exists  $s$  elements  $w_1, w_2, \dots, w_s$  such that there is no relation of length less than  $q$  among  $w_1, \dots, w_s$ . Let  $g_1, \dots, g_r$  be the standard generators of  $G_r$  and let  $w_i = W_i(g_1, \dots, g_r)$ ,  $1 \leq i \leq s$  where  $W_i$  is a reduced word in the free group of rank  $r$  formally generated by the letters  $g_1, \dots, g_r$ .

Since  $\Gamma$  is non-solvable, by Corollary 1.3 and Lemma 2.8. in [Bl], there exists an ordered  $(r+2)$ -tuple  $(f_0, f_1, \dots, f_r, f_{r+1})$  of elements of  $\Gamma$  which form a suitable tower of height  $r+2$ ; moreover, for all  $\delta > 0$ , there exists a tower  $(f'_1, \dots, f'_r, f'_{r+1})$  of height  $r+1$  such that  $\text{base}(f'_1)' \subseteq \text{base}(f_0)$  and the length of  $\text{base}(f'_1)$  is less than  $\delta$ .

Then for any  $\epsilon > 0$ , there exists a suitable tower  $(f_1^{(\epsilon)}, \dots, f_r^{(\epsilon)}, f_{r+1}^{(\epsilon)})$  in  $\Gamma$  such that  $\text{supp} f_i^{(\epsilon)} \subset (0, \epsilon)$ , for all  $i \in \{1, \dots, r+1\}$ ; moreover, we can find  $\epsilon_0 > 0$ , and a suitable tower  $(h_1, \dots, h_r, h_{r+1})$  of elements of  $\Gamma$  with bases  $(a_i, b_i) \subset (0, \epsilon_0)$ ,  $1 \leq i \leq r+1$  such that the following conditions hold:

- (i) for all  $\beta \in B_m(1)$ ,  $\beta(H) \subset (0, \epsilon_0)$ ;
- (ii) for any two distinct  $c_1, c_2 \in \mathbb{R}_+$ , and for all  $\beta_1 \in S_m(\Gamma, S, c_1)$ ,  $\beta_2 \in S_m(\Gamma, S, c_1)$ ,  $\beta_1(H) \cap \beta_2(H) = \emptyset$ ;
- (iii) for all  $c \in \mathbb{R}_+$ ,  $\beta_1, \beta_2 \in S_m(\Gamma, S, c)$ , and for all  $x \in H$ ,  $\beta_1(x) = \beta_2(x)$ . [so, in particular, for all  $\beta \in S_m(\Gamma, S)$  and for all  $x \in H$ ,  $\beta(x) = x$ ];

where  $H = \cup_{1 \leq i \leq r+1} \text{supp}(h_i)$ .

Let  $S^{(m)} = \{u_1 \gamma_1 u_1, \dots, u_s \gamma_s u_s, v_1, \dots, v_s\}$  where  $v_i = W_i(h_1, \dots, h_r)$ ,  $u_i = v_i^m$ ,  $1 \leq i \leq s$ .

Then there is no relation of length less than  $m$  among the elements of  $S^{(m)}$  because if  $W$  represents any such word in  $PL_o(I)$  then for some word  $U(h_1, \dots, h_r)$  and for all  $x \in U(a_{r+1}, b_{r+1})$  we have  $W(x) \notin U(a_{r+1}, b_{r+1})$  thus  $W(x) \neq x$ .  $\square$

**Remark 2.** As a corollary of Theorem 1, we obtain that  $\text{girth}(F) = \infty$ . This fact has been proved in [Br1] and in [AST]; both proofs use different ideas from each other and from the proof of Theorem 1. Theorem 1 also implies that  $\text{girth}(B) = \infty$  where  $B$  is the Brin group introduced in [Br2].

## GIRTH OF GROUPS WITH HYPERBOLIC-LIKE ELEMENTS

In this section we will present results about the girth of certain subgroups of  $\text{Homeo}(I)$ .

**Theorem 2.** Let  $\Gamma$  be any finitely generated subgroup of  $\text{Homeo}_+(I)$ . Assume that for any  $N \in \mathbb{N}$ , for any sequence  $0 < x_1 < x_2 < \dots < x_N < 1$ , and for any  $\epsilon > 0$ , one can find an element  $\gamma \in \Gamma$  such that  $\text{Fix}(\gamma) = \{0, c_1, \dots, c_N, 1\}$ , and  $|c_i - x_i| < \epsilon$ , for all  $1 \leq i \leq N$ . Then  $\text{girth}(\Gamma) = \infty$ .

**Remark 3.** As a corollary of Theorem 2, we obtain yet another proof of the fact that  $\text{girth}(F) = \infty$ .

**Proof of Theorem 2.** Let  $d(\Gamma) = s$ , and  $m$  be a natural number. We will find  $d+1$  generators of  $\Gamma$  such that there is no relation of length  $m$  or less in  $\Gamma$  in these generators. (since  $m$  is arbitrary, this proves that  $\text{girth}(\Gamma) = \infty$ ).

Let  $S = \{X_1, \dots, X_s\}$  be a generating set of  $\Gamma$ ,  $S^* = \{X_1, \dots, X_s, X_1^{-1}, \dots, X_s^{-1}\}$ . Let also  $I_0 = (\frac{1}{3}, \frac{2}{3})$ . We can find a natural number  $N > 4m$  and numbers  $0 = c_0 < c_1 < c_2 < \dots < c_N < c_{N+1} = 1$  such that

- (i)  $I_0 \subset (c_{2m}, c_{N-2m})$ .
- (ii)  $W(X_1, \dots, X_s)(I_0) \subset (c_1, c_N)$  for all non-trivial reduced words  $W$  of length at most  $2m$ .
- (iii)  $X(c_i) \notin \{c_1, \dots, c_N\}$  for all  $1 \leq i \leq N$  and for all  $X \in S^*$ .
- (iv)  $X(c_i) \notin [c_{i-1}, c_{i+1}]$  for all  $1 \leq i \leq N$  and for all  $X \in S^*$ .

Then we can find  $\epsilon > 0$ , a natural number  $M > 2m$  and an element  $\gamma \in \Gamma$  such that

- (v)  $\epsilon < \frac{c_{i+1} - c_i}{2}$ , for all  $0 \leq i \leq N$ .
- (vi)  $Fix(\gamma) = \{0, c_1, c_2, \dots, c_N, 1\}$
- (vii) for all  $n \geq M$ , we have  $\gamma^{\pm n}(U) \subset V$  where  $U = \sqcup_{1 \leq i \leq N-1} (c_i + \epsilon, c_{i+1} - \epsilon)$ ,  $V = \sqcup_{1 \leq i \leq N} (c_i - \epsilon, c_i + \epsilon)$ .
- (viii) for all  $n \geq M$  and for all  $X \in S^*$  we have  $X(V) \subset U$ .
- (ix)  $I_0 \cap U \neq \emptyset$ .

It is straightforward to make all these arrangements. [for condition (vii): we may simply arrange that  $\gamma^n(c_i + \epsilon, c_{i+1} - \epsilon) \subset (c_{i+1} - \epsilon, c_{i+1} + \epsilon)$  and  $\gamma^{-n}(c_i + \epsilon, c_{i+1} - \epsilon) \subset (c_i - \epsilon, c_i + \epsilon)$  for all  $1 \leq i \leq N - 1$ , and for all  $n \geq M$ ];

Now let  $p \geq 2M, S' = \{\gamma, \gamma^p X_1 \gamma^p, \dots, \gamma^{3p} X_2 \gamma^{3p}, \dots, \gamma^{(2s+1)p} X_s \gamma^{(2s+1)p}\}$ .

Then if  $W_0$  is a non-trivial reduced word in these generators of length at most  $m$  and  $x \in I_0 \cap U$ , and if  $W'$  is any suffix of  $W$  in the alphabet  $S$ , then because of (i), (ii), (iv)-(viii), we have  $W'(x) \in (c_1, c_N)$ . Then clearly (ping-pong argument)  $W_0(x) \in V$ , hence  $W_0(x) \neq x$ , hence  $W_0 \neq 1$ .  $\square$

**Remark 4.** The assumptions of Theorem 2. can be weakened significantly (at the expense of making the statement more technical).

**Remark 5.** From the proof we see that one can state a much more general theorem for the girth of groups acting on metric spaces by homeomorphisms. For every non-elementary word hyperbolic group we do have such a theorem indeed (see Theorem 1, [Ak2], where the metric space is the boundary of the group, and for every hyperbolic element we have one attracting and one repelling point). In our case, the metric space is typically non-compact [in the case of Theorem 2, the metric space is the non-compact space  $(0, 1) \cong \mathbb{R}$ ], and the "hyperbolic-like" elements have several points (instead of two) which are "attractive-repelling like" within "certain compact subspace".

We would like to give a precise definition of a hyperbolic-like element

**Definition 4.** Let  $X$  be a locally compact Hausdorff space,  $\Gamma$  be a subgroup of  $Homeo(X)$ ,  $S \subseteq \Gamma$ . Given an element  $\gamma \in \Gamma$  and a compact subset  $K \subseteq X$ , we say  $\gamma$  is *hyperbolic-like relative to  $K$*  if there exists a finite subset  $K_0 = \{x_1, \dots, x_n\} \subseteq X$  such that for all open non-empty neighborhoods  $U_1, \dots, U_n$  of  $x_1, \dots, x_n$  respectively, there exists a natural number  $M$  such that for all  $m > M$ ,  $\gamma^{\pm m}(K \setminus \cup_{1 \leq i \leq n} U_i) \subseteq \cup_{1 \leq i \leq n} U_i$ . We say  $\gamma$  is an  *$S$ -generic hyperbolic-like element relative to  $K$*  if also the following condition holds:  $s(x_i) \neq x_j$ , for all  $s \in S \cup S^{-1}, 1 \leq i \leq j \leq n$ .

**Remark 6.** A group  $\Gamma \leq \text{Homeo}(I)$  can be also viewed as a subgroup of homeomorphisms of  $X = (0, 1) \cong \mathbb{R}$ . Then, for  $\gamma \in \Gamma$ , if  $\text{Fix}(\gamma)$  is finite then  $\gamma$  is hyperbolic-like with respect to  $[\min \text{Fix}(\gamma), \max \text{Fix}(\gamma)]$ .

The following theorem is now clear from the proof of Theorem 2.

**Theorem 3.** Let  $X$  be a locally compact Hausdorff space,  $\Gamma$  be a finitely generated subgroup of  $\text{Homeo}(X)$ , and  $S$  be a finite generating set of  $\Gamma$ . Assume that for all compact subsets  $K$  of  $X$ , there exists an  $S$ -generic hyperbolic-like element of  $\Gamma$  relative to  $K$ . Then  $\text{girth}(\Gamma) = \infty$ .  $\square$

**Remark 7.** Theorem 3. generalises Theorem 2.1. from [Ak2] which states that any finitely generated non-elementary subgroup of a word hyperbolic group has infinite girth. In [Y], S.Yamagata proves another generalizations of Theorem 2.1. for the so-called convergence groups.

**Remark 8.** It is interesting that the group  $F$  (the standard representation of it in  $PL_o(I)$ ) is very rich with hyperbolic-like elements; for example, for any finite generating set  $S$  of  $F$ , and for any compact subset  $K_\epsilon = [\epsilon, 1 - \epsilon]$ ,  $\epsilon \in (0, 1)$ , there exists an  $S$ -generic hyperbolic-like elements relative to  $K_\epsilon$ . It is not known to us if the same is true for the standard (or any) representation of the Brin group  $B$ , [Br2].

We will borrow the following definition from [Ak3]

**Definition 5.** Let  $\Gamma$  be a finitely generated group,  $d(\Gamma)$  be the minimal number of a generating set of  $\Gamma$  and  $k \geq d(\Gamma)$  be a positive integer. Then we define  $\text{girth}_k(\Gamma) = \sup_{\langle S \rangle = \Gamma, |S| \leq k} \text{girth}(\Gamma, S)$ .

While proving Theorem 1, we indeed proved a bit more, namely, for any non-solvable finitely generated group  $\Gamma$  of  $PL_o(I)$ , we proved that  $\text{girth}_{2d}(\Gamma) = \infty$  where  $d = d(\Gamma)$ . But it is easy to modify the proof slightly to show that  $\text{girth}_{d+2}(\Gamma) = \infty$ . Also, in the proof of Theorem 2, we indeed prove that  $\text{girth}_{|S|+1}(\Gamma) = \infty$ , and one can prove the same in Theorem 3.

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